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Quasicomplements

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It will be shown that if X_1, X_2 and Y are subspaces of a separable normed linear space T for which $X_1 \subset X_2$, $\mathrm{Cl}\,(X_2 + Y)$ has infinite codimension, and $d(X_1, S_H) = 0$ if S_H is the unit sphere of a subspace H of Y with finite codimension in Y, then for any positive ϵ there is a subspace $\tilde{Y}$ of T such that Y is ϵ-near to $\tilde{Y}$ and $\tilde{Y}$ is a quasicomplement for all subspaces Z such that $X_1 \subset Z \subset \mathrm{Cl}\,(X_2 + Y)$. If X and Y are subspaces of T for which $\bar{X}$ has infinite codimension, then X has a quasicomplement $\tilde{Y}$ for which either $\bar{Y}$ is ϵ-near to a complemented subspace of $\tilde{Y}$ or $\bar{X} \cap \bar{Y}$ is ϵ-near to $\tilde{Y}$ (Theorem 2). If X and Y are closed quasicomplements but not complements in a separable reflexive Banach space, there exist subspaces Y_1 and Y_2 such that $Y_1 \subset Y \subset Y_2$, Y_1 is of infinite codimension in Y, Y has infinite codimension in Y_2, and Z is a quasicomplement for X if $Y_1 \subset Z \subset Y_2$. The existence of Y_1 does not depend on reflexivity.

By a subspace X of a normed linear space T being *ϵ-near* to a subspace Y we shall mean that there is an isomorphism L of X onto Y with

$$\| x - L(x) \| \leqslant \epsilon \| x \| \qquad \text{for all } x. \tag{1}$$

It is instructive to note that if X is ϵ-near to Y, then it follows from (1) that $\| L(x) \| \leqslant (1 + \epsilon) \| x \|$. Also,

$$\| x \| \leqslant \| x - L(x) \| + \| L(x) \| \leqslant \epsilon \| x \| + \| L(x) \|,$$

so that $(1 - \epsilon) \| x \| \leqslant \| L(x) \|$. Thus X and Y are *ϵ-isometric* in the sense described in [3], i.e.,

$$(1 - \epsilon) \| x \| \leqslant \| L(x) \| \leqslant (1 + \epsilon) \| x \|.$$

A *complement* for a subspace X of a normed linear space T is a subspace Y for which $X + Y = T$ and $\bar{X} \cap \bar{Y} = \{0\}$. A *quasicomplement* for a subspace X of a normed linear space T is a subspace Y for which $X + Y$ is dense in T and $\bar{X} \cap \bar{Y} = \{0\}$. If T is complete, then quasicomplements X and Y are

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complements if and only if there is an $\epsilon > 0$ for which $\| x - y \| > \epsilon \| x \|$ if $x \in X$ and $y \in Y$.

It was shown by Murray that all subspaces of reflexive separable Banach spaces have quasicomplements [6, p. 93, Corollary]. He also showed that if X and Y are quasicomplements that are not complements, then X has quasicomplements Y_1 and Y_2 for which $Y_1 \subset Y \subset Y_2$ and neither Y_1 nor Y_2 is Y [6, p. 94, Theorem].

The theorem of Murray was generalized by Mackey, who showed that all closed subspaces of a separable normed linear space have quasicomplements [5, p. 322]. Lindenstrauss has shown that if X is a closed subspace of a Banach space B, then X has a quasicomplement in B if both X and B are w-compactly generated [4], i.e., if there is a w-compact set S whose closed linear span is X (or B). However, there exist Banach spaces with subspaces that have no quasicomplements [4], although all subspaces of l^∞ have quasicomplements [7].

Guarii and Kadec proved that if X and Y are infinite-dimensional subspaces of a separable Banach space B for which $X \subset Y$ and Y has infinite codimension, then Y has a quasicomplement $\tilde{X}$ that is ϵ-isometric to X, and also X has a quasicomplement $\tilde{Y}$ that is ϵ-isometric to Y [3, p. 968, Theorem 4]. It follows from this theorem that any infinite-dimensional subspace of a separable Banach space has a quasicomplement with a basis and that any separable Banach space has subspaces X and Y that are quasicomplements and have bases [3, p. 968, Theorem 5 and Corollary 2]. Moreover, if $\epsilon > 0$ and X is an infinite-dimensional subspace of a separable Banach space B in which X has infinite codimension, then X has a quasicomplement Y that is ϵ-isometric (and ϵ-near) to X. Also, any separable Banach space that contains a reflexive subspace (or a subspace isomorphic to Hilbert space) is the closed linear span of two reflexive spaces with bases (or two subspaces isomorphic to Hilbert space); if the space is nonreflexive, it also is the closed linear span of two nonreflexive subspaces with bases.

Suppose X is an infinite-dimensional subspace of a separable Banach space B in which X has infinite codimension. The argument to be employed in Theorem 1 via biorthogonal sequences can be illustrated simply by the following outline of the proof via bases that if $Y \subset \bar{X}$, if $\bar{X}$ has infinite codimension, and if $\bar{Y}$ has a basis $\{y_n\}$, then X has a quasicomplement $\tilde{Y}$ that is isomorphic to Y. It is well known that $\{y_n\}$ is a basis for its closed linear span if and only if there is an $\epsilon > 0$ for which

$$\left\| \sum_1^{n+p} a_i y_i \right\| \geqslant \epsilon \left\| \sum_1^n a_i y_i \right\|$$

for all n, p, and $a_1, a_2, \ldots, a_{n+p}$ [1, p. 111; 2]. If $\sum_1^\infty \| \omega_n \| / \| y_n \| < \frac{1}{2}\epsilon$, then

$\{y_n + \omega_n\}$ is a basis for its closed linear span and $\sum a_i y_i \to \sum a_i(y_i + \omega_i)$ defines an isomorphism of Y onto a subspace $\tilde{Y}$ of Cl [lin $\{y_n + \omega_n\}$]. Now choose $\{\omega_n\}$ so that lin $\{\omega_n\}$ is dense in B, $\sum_1^\infty \|\omega_n\|/\|y_n\| < \frac{1}{2}\epsilon$, and

(a) $d(\omega_n, \text{lin}\{\bar{X}, \omega_1, ..., \omega_{n-1}\}) = \epsilon_n > 0$,

(b) $\sum_{n+1}^{\infty} \|\omega_i\| < \frac{1}{2}\epsilon_n$.

Let $Z = \text{Cl}[\text{lin}\{y_n + \omega_n\}]$. Then $\bar{X} + Z$ contains each ω_n and therefore is dense in B, so that $X + \tilde{Y}$ is dense in B. If $u \in \bar{X} \cap Z$ and $u \neq 0$, then

$$u = \sum_1^\infty a_i(y_i + \omega_i).$$

Since $\bar{Y} \subset \bar{X}$, it follows that $\sum_1^\infty a_i\omega_i \in \bar{X}$. If n is chosen so that $|a_n| > \frac{1}{2}\sup|a_i|$, then it follows from (a) that

$$\left\|\sum_{n+1}^{\infty} a_i\omega_i\right\| = \left\|a_n\omega_n - \left(\sum_1^\infty a_i\omega_i - \sum_1^{n-1} a_i\omega_i\right)\right\| > \tfrac{1}{2}\epsilon_n \sup|a_i|.$$

This is contradictory to the following inequality obtained from (b):

$$\left\|\sum_{n+1}^{\infty} a_i\omega_i\right\| \leqslant \tfrac{1}{2}\epsilon_n \sup|a_i|.$$

Before stating the principal theorem, certain more-or-less known facts are summarized for completeness in the next two lemmas. A *biorthogonal sequence* is a sequence $\{(y_n, g_n)\}$ for which $g_i(y_j) = \delta_j^i$; it is *normalized* if $\|y_n\| = 1$ for each n; it is *fundamental* for Y if lin $\{y_n\}$ is dense in Y; and it is *total* for Y if $y = 0$ whenever $g_n(y) = 0$ for all y.

LEMMA 1. *Let Y be a separable subspace of a normed linear space T and $\{(y_n, g_n)\}$ be a fundamental biorthogonal sequence for Y. If $0 < \epsilon < \frac{1}{2}$, if*

$$\sum_1^\infty \|g_i\| \|\omega_i\| < \epsilon, \tag{2}$$

and if $\tilde{Y} = \text{Cl}[\text{lin}\{y_n + \omega_n\}]$, then

(i) $\|\tilde{g}_k\| \leqslant \|g_k\|/(1-\epsilon) \leqslant \|\tilde{g}_k\|/(1-2\epsilon)$, *where $\tilde{g}_k$ is the continuous linear functional on $\tilde{Y}$ for which $\tilde{g}_k(y_j + \omega_j) = \delta_k^j$;*

(ii) *$\{g_i\}$ is total for $\bar{Y}$ if and only if $\{\tilde{g}_i\}$ is total for $\tilde{Y}$;*

(iii) *$\bar{Y}$ is ϵ-near to $\tilde{Y}$.*

Proof. To prove (i), note that if $1 \leqslant k \leqslant n$, then

$$| a_k | \leqslant \| g_k \| \left\| \sum_1^n a_i y_i \right\|. \tag{3}$$

It follows from this and (2) that

$$\left\| \sum_1^n a_i(y_i + \omega_i) \right\| \geqslant \left\| \sum_1^n a_i y_i \right\| - \sum_1^n | a_i | \| \omega_i \|$$

$$\geqslant \left\| \sum_1^n a_i y_i \right\| - \left\| \sum_1^n a_i y_i \right\| \sum_1^\infty \| g_i \| \| \omega_i \|$$

$$\geqslant (1 - \epsilon) \left\| \sum_1^n a_i y_i \right\| \geqslant (1 - \epsilon) | a_k | / \| g_k \|.$$

Thus $| \tilde{g}_k [\sum_1^n a_i (y_i + \omega_i)] | = | a_k | \leqslant \| \sum_1^n a_i (y_i + \omega_i) \| \| g_k \| / (1 - \epsilon)$ and

$$\| \tilde{g}_k \| \leqq \frac{\| g_k \|}{1 - \epsilon}. \tag{4}$$

From this, $\sum_1^\infty \| \tilde{g}_i \| \| \omega_i \| < \epsilon/(1 - \epsilon)$, so that it follows from (4) that

$$\| g_k \| \leqslant \frac{\| \tilde{g}_k \|}{1 - \epsilon/(1 - \epsilon)} \quad \text{or} \quad \| g_k \| \leqslant \frac{1 - \epsilon}{1 - 2\epsilon} \| \tilde{g}_k \|.$$

To prove (ii), suppose that $\{\tilde{g}_n\}$ is not total for $\tilde{Y}$, so that there is a u with $\| u \| = 1$ that belongs to Cl [lin $\{y_i + \omega_i : i \geqslant n\}$] for all n. Then for any $\delta > 0$ there is a finite sum for which

$$\| u - \textstyle\sum_n b_i (y_i + \omega_i) \| < \delta,$$

$$| b_i | < \delta \| \tilde{g}_i \| + | \tilde{g}_i(u) | \leqslant (1 + \delta) \| \tilde{g}_i \| \leqslant (1 + \delta) \| g_i \| / (1 - \epsilon), \quad \text{and}$$

$$\| u - \textstyle\sum_n b_i y_i \| < \delta + \| \textstyle\sum_n b_i \omega_i \|$$

$$< \delta + \frac{1 + \delta}{1 - \epsilon} \sum_n \| g_i \| \| \omega_i \|.$$

Since δ was arbitrary and the last summation approaches zero as n increases, it follows that $u \in$ Cl [lin $\{y_i : i \geqslant n\}$] for all n and $\{g_i\}$ is not total for $\bar{Y}$. It follows by a similar argument that $\{\tilde{g}_i\}$ is not total for $\tilde{\bar{Y}}$ if $\{g_i\}$ is not total for $\bar{Y}$.

To prove (iii), define a linear map L from lin $\{y_n\}$ onto lin $\{y_n + \omega_n\}$ by letting

$$L\left(\sum_1^n a_i y_i\right) = \sum_1^n a_i(y_i + \omega_i).$$

It follows from (2) and (3) that

$$\left\| \sum_1^n a_i y_i - \sum_1^n a_i(y_i + \omega_i) \right\| = \left\| \sum_1^n a_i \omega_i \right\| \leqslant \epsilon \cdot \sup \frac{|a_i|}{\|g_i\|} \leqslant \epsilon \left\| \sum_1^n a_i y_i \right\|.$$

Therefore, for all y in $\operatorname{lin}\{y_i\}$, $\|y - L(y)\| \leqslant \epsilon \|y\|$. If L is extended to be continuous on $\bar{Y}$, then L and L^{-1} are continuous and the range of L is $\tilde{Y}$.

LEMMA 2. *Let Y be a separable normed linear space and suppose that to each subspace H of finite codimension in Y there is associated a nonempty subset P_H of the unit sphere of H. Then Y has a normalized biorthogonal sequence $\{(y_n, g_n)\}$ that has the properties*:

(i) *$\{y_n\}$ is fundamental and $\{g_n\}$ is total for Y*;

(ii) *$y_{3k+1} \in P_H$, where $H = \{y : g_i(y) = 0$ for $i < 3k + 1\}$*;

(iii) *$\|g_n\| \leqslant 2^{n-1}$ for each n.*

Proof. Let $\{\eta_n\}$ be dense in the unit sphere of Y and $\{U_n\}$ be a sequence of balls with radii 1/6 and η_n the center of U_n for each n. Choose $\{(y_n, f_n)\}$ inductively so that $\|y_n\| = \|f_n\| = f_n(y_n) = 1$ for each n and

(a) $f_i(y_n) = 0$ if $i < n$;

(b) $y_{3k+1} \in P_H$, where $H = \{y : f_i(y) = 0$ for $i < 3k + 1\}$,

(c) $\eta_{k+1} \in \operatorname{lin}\{y_1, y_2, \ldots, y_{3k+2}\}$ if $k \geqq 0$,

(d) $\|y_{3k+3} - \eta_{k+1}\| < 1/3$ if U_{k+1} has any nonzero member y for which $f_i(y) = 0$ if $i \leqq 3k + 2$.

Whatever the choices of $f_1, \ldots, f_{n-1}$, there is a y_n of unit norm that satisfies (a) and an f_n with $\|f_n\| = f_n(y_n) = 1$; by assumption, condition (b) can be satisfied; (c) can be satisfied by letting y_{3k+2} be a linear combination of $\{y_1, \ldots, y_{3k+1}, \eta_{k+1}\}$ if $\eta_{k+1} \notin \operatorname{lin}\{y_1, \ldots, y_{3k+1}\}$; a little computation shows that (d) is satisfied if $y_{3k+3} = y/\|y\|$.

It follows from (c) that $\{y_n\}$ is fundamental. Let $\{g_n\}$ be defined so that $g_i(y_j) = \delta_j^i$. If $\{g_n\}$ is not total, there is a y with $\|y\| = 1$ for which $y \in \mathrm{Cl}[\operatorname{lin}\{y_i : i \geqslant n\}]$ for all n. Then $f_n(y) = 0$ for all n, so that it follows from (d) and the density of $\{\eta_n\}$ that there is a y_{3k+3} for which $\|y - y_{3k+3}\| < \frac{1}{2}$. Then $|f_{3k+3}(y - y_{3k+3})| < \frac{1}{2}$ and $f_{3k+3}(y_{3k+3}) = 1$, which imply that $|f_{3k+3}(y)| > \frac{1}{2}$.

Now suppose there is an n for which $\|g_n\| > 2^{n-1}$. Then there are numbers $a_1, \ldots, a_{n-1}$ and an h such that $f_i(h) = 0$ if $i \leqslant n$ and

$$\left\| y_n - \left(\sum_1^{n-1} a_i y_i + h \right) \right\| < 2^{1-n}.$$

By using f_1, we obtain $|a_1| < 2^{1-n}$. If $|a_i| < 2^{i-n}$ for $i < k < n$, then it follows from $|\sum_1^{k-1} a_i f_k(y_i) + a_k| < 2^{1-n}$ that

$$|a_k| < 2^{1-n} + \sum_{i=1}^{k-1} 2^{i-n} = 2^{k-n},$$

so that $|a_i| < 2^{i-n}$ for $i < n$. Therefore, it follows from

$$\left| f_n(y_n) - \sum_1^{n-1} a_i f_n(y_i) \right| < 2^{1-n}$$

that $1 < 2^{1-n} + \sum_1^{n-1} |a_i| < 2^{1-n} + \sum_{i=1}^{n-1} 2^{i-n} = 1$, which completes the proof.

If X and Y are subspaces of a normed linear space and $X \cap Y$ is infinite dimensional, then the set P_H of the preceding lemma could be the unit sphere of $X \cap H$. In the next theorem, this is generalized somewhat so that P_H is merely close to X.

THEOREM 1. *Let X_1, X_2, and Y be subspaces of a separable normed linear space T for which $X_1 \subset X_2$, $\mathrm{Cl}(X_2 + Y)$ has infinite codimension, and $d(X_1, S_H) = 0$ if S_H is the unit sphere of a subspace H of Y with finite codimension in Y. Then for any positive ϵ there is a subspace $\tilde{Y}$ of T such that Y is ϵ-near to $\tilde{Y}$ and $\tilde{Y}$ is a quasicomplement for all subspaces Z such that*

$$X_1 \subset Z \subset \mathrm{Cl}(X_2 + Y).$$

Proof. There is no loss of generality if we assume that $\epsilon < \frac{1}{2}$. Let $\{U_n\}$ be a sequence of balls with radii less than 1 and centers on the unit sphere of T for which $\mathrm{lin}\{u_n\}$ is dense in T if $u_n \in U_n$ for each n. Let ρ_n be the radius of U_n and let $\{\beta_n\}$ be a sequence of positive numbers for which

(a) $\sum_1^\infty 2^{i-1}\beta_i < \epsilon$,

(b) $2^{i-1}\beta_i < \rho_n\beta_n (1 - 2\epsilon)/(128 \cdot 2^{i-n})$ if $i > n$.

If H is a subspace of Y with codimension $3k$, let P_H of Lemma 2 be

$$P_H = \{h : \|h\| = 1 \quad \text{and} \quad d(h, X_1) < \tfrac{1}{8}\rho_k\beta_{3k+1}\}.$$

It then follows from Lemma 2 that there is a normalized biorthogonal sequence $\{(y_n, g_n)\}$ for which $\{y_n\}$ is fundamental and $\{g_n\}$ is total for Y, $d(y_{3k+1}, X_1) < \frac{1}{8}\rho_k\beta_{3k+1}$, and $\|g_n\| \leqslant 2^{n-1}$ for all n. Choose a sequence $\{\omega_n\}$ inductively so that

(c) $\frac{1}{2}\beta_n < \|\omega_n\| < \beta_n$ for each n,

(d) $d(\omega_n, \mathrm{lin}\{X_2 + Y, \omega_1, \ldots, \omega_{n-1}\}) > \frac{1}{8}\rho_n\|\omega_n\|$ for all n,

(e) for each k, there is an x_{3k+1} in X_1 and μ_{3k+1} in T such that some scalar multiple of μ_{3k+1} belongs to U_{3k+1} and

$$y_{3k+1} + \omega_{3k+1} = x_{3k+1} + \mu_{3k+1}\,.$$

Clearly (c) and (d) can be satisfied for all n, if these and (e) can be satisfied for $n = 3k + 1$. Suppose that $\omega_1, \ldots, \omega_{3k}$ have been chosen and let $V = \text{lin}\{X_2 + Y, \omega_1, \ldots, \omega_{3k}\}$. Since the center of U_n has unit norm, $\| u_n \| \leqslant 1 + \rho_n < 2$ if $u_n \in U_n$. Choose u_n so that

$$\| u_n - v \| > \tfrac{2}{3}\, \rho_n > \tfrac{1}{3}\, \rho_n \| u_n \| \quad \text{if} \quad v \in V. \tag{5}$$

Let

$$\mu_{3k+1} = \frac{3}{4}\, \beta_{3k+1} \frac{u_{3k+1}}{\| u_{3k+1} \|}\,. \tag{6}$$

Choose an x_{3k+1} in X_1 for which

$$\| y_{3k+1} - x_{3k+1} \| < \tfrac{1}{8}\, \rho_k \beta_{3k+1}\,. \tag{7}$$

If $\omega_{3k+1} = x_{3k+1} + \mu_{3k+1} - y_{3k+1}$, then (e) is satisfied and

$$\| \mu_{3k+1} \| - \| x_{3k+1} - y_{3k+1} \| \leqslant \| \omega_{3k+1} \| \leqslant \| \mu_{3k+1} \| + \| x_{3k+1} - y_{3k+1} \|,$$

so that it follows from (6), (7), and $\rho_{3k+1} < 1$ that

$$\tfrac{1}{2}\, \beta_{3k+1} < \| \omega_{3k+1} \| < \beta_{3k+1} \tag{8}$$

and (c) is satisfied. To show that (d) is satisfied, note first that if $v \in V$ then

$$\begin{aligned} \| \omega_{3k+1} - v \| &= \| x_{3k+1} + \mu_{3k+1} - y_{3k+1} - v \| \\ &\geqslant \| \mu_{3k+1} - v \| - \| x_{3k+1} - y_{3k+1} \|. \end{aligned}$$

It now follows from (5) and (7), and then (6) and (7), that

$$\begin{aligned} \| \omega_{3k+1} - v \| &\geqslant \tfrac{1}{3}\, \rho_{3k+1} \| \mu_{3k+1} \| - \tfrac{1}{8}\, \rho_{3k+1}\beta_{3k+1} \\ &= \tfrac{1}{8}\, \rho_{3k+1}\beta_{3k+1} > \tfrac{1}{8}\, \rho_{3k+1} \| \omega_{3k+1} \|, \end{aligned}$$

so that (d) is satisfied.

Let $\tilde{Y} \subset \text{Cl}[\text{lin}\{y_n + \omega_n\}]$ be the image of Y for the linear map L defined by letting $L(\sum_1^n a_i y_i) = \sum_1^n a_i(y_i + \omega_i)$ and extending L to Y. Since

$$\sum_1^\infty \| g_i \| \| \omega_i \| < \sum_1^\infty 2^{i-1}\beta_i < \epsilon,$$

it follows from (iii) of Lemma 1 that Y is ϵ-near to $\tilde{Y}$. It follows from (e) and the choice of the spheres $\{U_n\}$ that $X_1 + \tilde{Y}$ is dense in T, and from this that $Z + \tilde{Y}$ is dense in T if $X_1 \subset Z \subset \mathrm{Cl}\,(X_2 + Y)$. To complete the proof, we need only show that $\mathrm{Cl}\,(X_2 + Y) \cap \mathrm{Cl}(\tilde{Y}) = \{0\}$. Suppose that $v \in \mathrm{Cl}(X_2 + Y) \cap \mathrm{Cl}(\tilde{Y})$ and $\| v \| = 1$. For each i, let $\tilde{g}_i$ be the continuous linear functional on $\tilde{Y}$ for which $\tilde{g}_i(y_j + \omega_j) = \delta_j^i$ for all j. Then it follows from Lemma 1 that $\{\tilde{g}_n\}$ is total for $\mathrm{Cl}(\tilde{Y})$. Thus $0 < \sigma \leqslant 1$ if

$$\sigma = \sup_{i \geqslant 1} \frac{| \tilde{g}_i(v) |}{\| \tilde{g}_i \|}.$$

Choose n such that $| \tilde{g}_n(v) | > \frac{1}{2} \sigma \| \tilde{g}_n \|$ and then choose a finite sequence $\{b_i\}$ for which

$$\left\| v - \left[\sum_1^n \tilde{g}_i(v)(y_i + \omega_i) + \sum_{n+1} b_i(y_i + \omega_i) \right] \right\| < \kappa, \tag{9}$$

where κ is the smaller of $(1 - 2\epsilon)\, \sigma \rho_n \beta_n / [64(1 - \epsilon)]$ and σ. Then

$$\left\| \tilde{g}_n(v)\omega_n - \left[v - \sum_1^{n-1} \tilde{g}_i(v)(y_i + \omega_i) - \tilde{g}_n(v) y_n - \sum_{n+1} b_i y_i \right] \right\|$$
$$< \kappa + \sum_{n+1} \| b_i \omega_i \|.$$

Since $v \in \mathrm{Cl}\,(X_2 + Y)$, it follows from this inequality, (d) and the choice of n that

$$\frac{1}{16} \rho_n \sigma \| \tilde{g}_n \| \| \omega_n \| < \kappa + \sum_{n+1} \| b_i \omega_i \|.$$

If $i > n$, it follows from (9) that $| \tilde{g}_i(v) - b_i | < \kappa \| \tilde{g}_i \|$, so that

$$| b_i | < \kappa \| \tilde{g}_i \| + \sigma \| \tilde{g}_i \|,$$

and

$$\frac{1}{16} \rho_n \sigma \| \tilde{g}_n \| \| \omega_n \| < \kappa + (\kappa + \sigma) \sum_{n+1} \| \tilde{g}_i \| \| \omega_i \|.$$

Now it follows from (i) of Lemma 1, $\| g_n \| \| \omega_n \| > \frac{1}{2} \beta_n$, $\kappa \leqslant \sigma$, $\| g_i \| \leqslant 2^{i-1}$ and (8) that

$$\frac{1 - 2\epsilon}{32(1 - \epsilon)} \sigma \rho_n \beta_n < \kappa + \frac{2\sigma}{1 - \epsilon} \sum_{n+1} \| g_i \| \| \omega_i \| < \kappa + \frac{2\sigma}{1 - \epsilon} \sum_{n+1} 2^{i-1} \beta_i.$$

This and (b) imply that

$$\frac{1 - 2\epsilon}{32(1 - \epsilon)} \sigma \rho_n \beta_n < \kappa + \frac{1 - 2\epsilon}{64(1 - \epsilon)} \sigma \rho_n \beta_n.$$

Since $\kappa < (1 - 2\epsilon)\,\sigma\rho_n\beta_n/[64(1 - \epsilon)]$, we have a contradiction and it has been proved that $\mathrm{Cl}\,(X_2 + Y) \cap \mathrm{Cl}\,(\tilde{Y}) = \{0\}$.

COROLLARY 1. *Let X and Y be subspaces of a separable normed linear space T for which $X \cap Y$ is infinite dimensional and $\mathrm{Cl}(X + Y)$ has infinite codimension. Then for any positive ϵ there is a subspace $\tilde{Y}$ of T such that Y is ϵ-near to $\tilde{Y}$ and $\tilde{Y}$ is a quasicomplement for all Z such that*

$$X \cap Y \subset Z \subset \mathrm{Cl}(X + Y).$$

COROLLARY 2. *Let H be an infinite dimensional subspace of a normed linear space T and let W be any closed subspace of infiinite codimension in T that contains H. Then for any positive ϵ and any subspace Y with $H \subset Y \subset W$, there is a subspace $\tilde{Y}$ such that Y is ϵ-near to $\tilde{Y}$ and $\tilde{Y}$ is a quasicomplement for all Z such that $H \subset Z \subset W$.*

Proof. For Corollary 1, let $X \cap Y = X_1$ and $\mathrm{Cl}(X + Y) = X_2$ and apply Theorem 1. For Corollary 2, apply Theorem 1 again with $X_1 = H$ and $X_2 = W$.

The next theorem summarizes the types of quasicomplements of X that can be built from an arbitrary subspace Y.

THEOREM 2. *Let X and Y be subspaces of a normed linear space T for which $\bar{X}$ has infinite codimension. For any $\epsilon > 0$, X has a quasicomplement $\tilde{Y}$. If $\mathrm{Cl}(X + Y)$ has finite codimension, then $\tilde{Y}$ can be chosen, so that*

(i) *If $\bar{X} \cap \bar{Y}$ is finite dimensional, $\bar{Y}$ is ϵ-near to a complemented subspace of $\tilde{Y}$.*

Whatever the codimension of $\mathrm{Cl}(X + Y)$,

(ii) *If $\bar{X} \cap \bar{Y}$ [or $X \cap Y$] is infinite dimensional, then $\bar{X} \cap \bar{Y}$ is ϵ-near to $\tilde{Y}$ and $\tilde{Y}$ is a quasicomplement for all Z such that*

$$\bar{X} \cap \bar{Y} \subset Z \subset \bar{X} \text{ [or } X \cap Y \subset Z \subset \bar{X}].$$

If X is infinite dimensional and $\mathrm{Cl}(X + Y)$ has infinite codimension, then $\tilde{Y}$ can be chosen so that

(iii) *$\mathrm{Cl}(X + Y)$ is ϵ-near to $\tilde{Y}$ and $\tilde{Y}$ is a quasicomplement for all Z such that $X \subset Z \subset \mathrm{Cl}(X + Y)$; or*

(iv) *if $\bar{X} \cap \bar{Y}$ [or $X \cap Y$] is infinite-dimensional, then $\tilde{Y}$ can be chosen so that Y is ϵ-near to $\tilde{Y}$ and $\tilde{Y}$ is a quasicomplement for all Z such that*

$$\bar{X} \cap \bar{Y} \subset Z \subset \mathrm{Cl}(X + Y) \text{ [or } X \cap Y \subset Z \subset \mathrm{Cl}(X + Y)].$$

Proof. (i) If $\mathrm{Cl}(X + Y)$ has finite codimension and $\bar{X} \cap \bar{Y}$ is finite dimensional, there is a subspace Y_0 of $\bar{Y}$ which has finite codimension in $\bar{Y}$

and for which $\bar{X} \cap Y_0 = \{0\}$. Also, $\mathrm{Cl}(X + Y_0)$ has finite codimension in T, so that there is a finite-dimensional subspace V of T such that $Y_0 + V$ is a quasicomplement of X. If U is a finite-dimensional complement of Y_0 in $\bar{Y}$, then V can be chosen so that U is sufficiently near a subspace U_1 of V that $\bar{Y}$ is ϵ-near to $Y_0 + U_1$.

(ii) Let H of Corollary 2 be $\bar{X} \cap \bar{Y}$ (or $X \cap Y$) and let W be $\bar{X}$.

(iii) Let H of Corollary 2 be X and both W and Y be $\mathrm{Cl}(X + Y)$.

(iv) This is Corollary 1 applied to $\bar{X}$ and $\bar{Y}$ (or X and Y).

It was shown by Murray [6, p. 94, Theorem] that if X and Y are quasicomplements but not complements in a reflexive Banach space, then there are quasicomplements Y_1 and Y_2 for X such that $Y_1 \subset Y \subset Y_2$ and neither Y_1 nor Y_2 is equal to Y. Actually, the constructions used yield Y_1 of finite codimension in Y and Y of finite codimension in Y_2. Mackey notes [5, pp. 324–325] that reflexivity is not needed, but that completeness is critical for Murray's arguments. The next two theorems strengthen these results.

THEOREM 3. *Let X and Y be closed quasicomplements in a normed linear space T. If the closures of X and Y in the completion of T are not complements, then X has a closed quasicomplement Y_1 for which $Y_1 \subset Y$ and Y_1 has infinite codimension in Y.*

Proof. Since the closures of X and Y in the completion of T are not complements, $d(X, S_H) = 0$ if S_H is the unit sphere of a subspace H of finite codimension in Y. Therefore it follows from Lemma 2 that Y has a normalized biorthogonal sequence $\{(y_n, g_n)\}$ for which

(a) $\{y_n\}$ is fundamental and $\{g_n\}$ is total for Y,

(b) $d(y_{3k+1}, X) < (3k + 1)^{-1} \Pi_{i=2}^{3k+1} 2^{-2i}$,

(c) $\| g_n \| \leqslant 2^{n-1}$ for each n.

Let $p(n, i)$ be a one-to-one map of ordered pairs (n, i) of positive integers onto the set of positive integers such that for each n the sequence $\{p(n, i)\}$ contains infinitely many integers of type $3k + 1$. Let

$$\lambda_{p(n,i)} = 2^{-2p(n,i+1)}, \tag{10}$$

let Y_1 be the closed linear span of all $\lambda_{p(n,i)} y_{p(n,i)} - y_{p(n,i+1)}$, and Z_1 be the closed linear span of all $y_{p(n,1)}$. Clearly $\mathrm{Cl}(Y_1 + Z_1) = Y$.

To show that Y_1 has infinite codimension in Y, it is sufficient to show that $Y_1 \cap Z_1 = \{0\}$. Suppose $y \in Y_1 \cap Z_1$ and $\| y \| = 1$. Since $\{g_n\}$ is total for Y and $y \in Z_1$, there exist $a_1 \neq 0$ and n such that

$$y = a_1 \lambda_{p(n,1)} y_{p(n,1)} + h,$$

where $h \in \mathrm{Cl}[\mathrm{lin}\{y_i : i > p(n, 1)\}]$. Since $y \in Y_1$, there is a sequence of numbers $\{a_i\}$ such that, for any k, there is an h for which

$$\left\| y - \left(\sum_{i=1}^{k} a_i(\lambda_{p(n,i)} y_{p(n,i)} - y_{p(n,i+1)}) + h \right) \right\| < \frac{1}{4} | a_1 | 2^{1-p(n,2)}, \qquad (11)$$

where $h \in \mathrm{Cl}[\mathrm{lin}\{\lambda_{p(m,i)} - y_{p(m,i+1)} : m \neq n \text{ or } i > k\}]$. Since $y \in Z_1$, it follows that $g_{p(n,i+1)}(y) = 0$ if $i \geqslant 1$ and therefore that we can use (c) to obtain

$$| a_i - \lambda_{p(n,i+1)} a_{i+1} | < \tfrac{1}{4} | a_1 | 2^{p(n,i+1)-p(n,2)} \text{ for } 1 \leqslant i < k.$$

Now multiply by suitable λ's and add, to obtain

$$\left| a_1 - \left(\prod_{i=2}^{k} \lambda_{p(n,i)} \right) a_k \right|$$

$$< \tfrac{1}{4} | a_1 | 2^{-p(n,2)} \left(2^{p(n,2)} + 2^{p(n,3)} \lambda_{p(n,2)} + \cdots + 2^{p(n,k)} \prod_{i=2}^{k-1} \lambda_{p(n,i)} \right)$$

$$\leqslant \tfrac{1}{4} | a_1 | 2^{-p(n,2)} \left(2^{p(n,2)} + \sum_{i=3}^{k} 2^{-p(n,i)} \right)$$

$$< \tfrac{1}{4} | a_1 | \sum_{0}^{\infty} 2^{-i} = \tfrac{1}{2} | a_1 |,$$

so that

$$| a_k | > \tfrac{1}{2} | a_1 | \left[\prod_{i=2}^{k} \lambda_{p(n,i)} \right]^{-1} = \tfrac{1}{2} | a_1 | \prod_{i=2}^{k} 2^{2p(n,i+1)}. \qquad (12)$$

It follows from (c) and (10) that

$$\sum_{1}^{\infty} \lambda_i \| g_i \| \leqq \sum_{1}^{\infty} \lambda_i 2^{i-1} < \sum_{1}^{\infty} 2^{-2(i+1)} 2^{i-1} = \frac{1}{8}.$$

This together with (11) and (i) of Lemma 1 implies that

$$| a_k | \leqslant (1 + \tfrac{1}{4} | a_1 | 2^{1-p(n,2)}) \frac{8}{7} \| g_{p(n,k+1)} \| < (1 + \tfrac{1}{4} | a_1 |) \, 2^{p(n,k+1)}. \qquad (13)$$

Whatever the value of a_1, (12) and (13) are contradictory if k is sufficiently large.

To show that Y_1 is a quasicomplement for X, it is sufficient to show that $X + Y_1$ is dense in T. To do this, it is sufficient to show that $y_{p(n,1)}$ belongs to $\mathrm{Cl}(X + Y_1)$ for each n. Let $p(n, k)$ be of type $3p + 1$. It follows from (b) that there is an x in X for which

$$\| y_{p(n,k)} - x \| < (p(n, k))^{-1} \prod_{i=2}^{p(n,k)} 2^{-2i}.$$

Let

$$y = \sum_{j=1}^{k-1} \left(\prod_{i=1}^{j} \lambda_{p(n,i)} \right)^{-1} (\lambda_{p(n,j)} y_{p(n,j)} - y_{p(n,j+1)})$$

$$= y_{p(n,1)} - \left(\prod_{i=1}^{k-1} \lambda_{p(n,i)} \right)^{-1} y_{p(n,k)} \,.$$

Then $y \in Y_1$ and

$$\left\| y_{p(n,1)} - y - \left(\prod_{i=1}^{k-1} \lambda_{p(n,i)} \right)^{-1} x \right\| = \| y_{p(n,k)} - x \| \left(\prod_{i=1}^{k-1} \lambda_{p(n,i)} \right)^{-1}$$

$$= \| y_{p(n,k)} - x \| \prod_{i=2}^{k} 2^{2p(n,i)}$$

$$\leqslant \| y_{p(n,k)} - x \| \prod_{i=2}^{p(n,k)} 2^{2i}$$

$$< (p(n, k))^{-1}.$$

Since $p(n, k)$ can be arbitrarily large, this completes the proof of Theorem 3.

THEOREM 4. *Let X and Y be quasicomplements in a separable reflexive Banach space B. If $\overline{X}$ and $\overline{Y}$ are not complements for B, then X has a quasicomplement Y_2 for which $Y \subset Y_2$ and Y has infinite codimension in Y_2.*

Proof. Let X' and Y' be the sets of continuous linear functionals that are identically zero on $\overline{X}$ or $\overline{Y}$, respectively. Then $X' \cap Y' = \{0\}$, since $X + Y$ is dense in B. Also, $\mathrm{Cl}(X' + Y') = B^*$, since otherwise there would be a nonzero z in B such that $f(z) = 0$ if $f \in \mathrm{Cl}(X' + Y')$ and this would imply that $z \in \overline{X} \cap \overline{Y}$. Thus X' and Y' are quasicomplements in B^* (also see [6, p. 85, Corollary]). Let $\omega \in B \cap \sim(\overline{X} + \overline{Y})$. Then it is easy to see that $\overline{X}$ and $\mathrm{lin}(\overline{Y}, \omega)$ are quasicomplements for T, so that X' has a quasicomplement that is a proper subspace of Y'. This implies that X' and Y' are not complements and it then follows from Theorem 3 that there is a closed subspace Z'

of Y' that is of infinite codimension in Y' and such that Z' is a quasicomplement for X' in B^*. Let Y_2 be the set of all y in B with $f(y) = 0$ for all f in Z'. Since B is reflexive, Y has infinite codimension in Y_2. Since X' and Z' are quasicomplements for B^* and B is reflexive, it follows from the first part of this proof that X and Y_2 are quasicomplements for B.

It is not clear whether reflexivity is needed in Theorem 4. However, it seems reasonable to conjecture that the theorem is false without this hypothesis.

The next corollary follows from Corollary 1 and Theorems 3 and 4.

COROLLARY 3. *Let X and Y be subspace of a separable reflexive Banach space. If $X \cap Y$ is infinite dimensional and* $\mathrm{Cl}(X+Y)$ *has infinite codimension, then there are subspaces X_1, X_2, Y_1, $\tilde{Y}$ and Y_2 such that*

(i) *$X_1 \subset X \cap Y$ and X_1 has infinite codimension in $X \cap Y$,* $\mathrm{Cl}(X+Y) \subset X_2$ *and* $\mathrm{Cl}(X+Y)$ *has infinite codimension in X_2;*

(ii) *Y is ϵ-near to $\tilde{Y}$,*

(iii) *$Y_1 \subset \tilde{Y} \subset Y_2$, Y_1 has infinite codimension in $\tilde{Y}$, and $\tilde{Y}$ has infinite codimension in Y_2;*

(iv) *U and V are quasicomplements if $X_1 \subset U \subset X_2$ and $Y_1 \subset V \subset Y_2$.*

COROLLARY 4. *Let B be a separable Banach space that has an infinite dimensional subspace isomorphic to Hilbert space. Then for any $\epsilon > 0$ there are subspaces H_1, H_2, H_3, and H_4 isomorphic to Hilbert space and such that H_1 is ϵ-near to H_4, $H_1 \subset H_2$, $H_3 \subset H_4$, H_1 has infinite codimension in H_2, H_3 has infinite codimension in H_4, and subspaces X and Y of B are quasicomplements if*

$$H_1 \subset X \subset H_2 \text{ and } H_3 \subset Y \subset H_4 .$$

Proof. Let H_2 be any subspace that is isomorphic to Hilbert space and has infinite codimension in B, and let H_1 be any infinite-dimensional subspace of H_2 that has infinite codimension in H_2. It then follows from Corollary 2 that there is a subspace H_4 such that H_1 is ϵ-near to H_4 and H_4 is a quasicomplement for all X such that

$$H_1 \subset X \subset H_2 .$$

Theorem 3 gives a subspace H_3 of H_4 that has infinite codimension in H_4 and is a quasicomplement for H_1. If $H_1 \subset X \subset H_2$ and $H_3 \subset Y \subset H_4$, then $X \cap Y \subset H_2 \cap H_4 = \{0\}$. Since $X + Y \supset H_1 + H_3$, $X + Y$ is dense in B and X and Y are quasicomplements.

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